ON THE BJÖRLING PROBLEM IN A TRIDIMENSIONAL LIE GROUP

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Abstract. We prove existence and uniqueness of the solution of the Björling problem for minimal surfaces in a tridimensional Lie group.

1. Introduction.

The Weierstrass representation formula for minimal surfaces in $\mathbb{R}^3$ has been a fundamental tool for producing examples and to prove general properties of such surfaces, since it allows to bring into the problem the theory of holomorphic function of one complex variable. In ([2]) the authors describe a general Weierstrass representation formula for minimal surfaces in an arbitrary Riemannian manifold. The P.D.E. involved are, in general, too complicated to be solved explicitly. However, for particular ambient manifolds, as the Heisenberg group, the hyperbolic space and the product of hyperbolic plane with $\mathbb{R}$, the equations are more workable and the formula can be used to produce examples (see ([1]), ([2])).

In this note we will show how this formula can be used, at least if the ambient manifold is a 3-dimensional Lie group, to get general existence and uniqueness results, as the solution of the Björling's problem.

2. The Weierstrass representation formula.

The arguments will be essentially local so we will consider, as ambient manifold M, $\mathbb{R}^3$ with a Riemannian metric $g = (g_{ij})$. We will denote with $\Omega \subseteq \mathbb{C} \cong \mathbb{R}^2$ a simply connected domain with a complex coordinate $z = u + iv$, $u, v \in \mathbb{R}$. Also we will use the standard notations for complex derivatives:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right); \qquad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

In this situation, the general Weierstrass representation formula can be stated as follows:

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Theorem 2.1. *Let $f:\Omega\longrightarrow M$ be a conformal minimal immersion. The complex tangent vector:*

$$\frac{\partial f}{\partial z} := \phi := \sum \phi_i \frac{\partial}{\partial x_i}, \qquad \phi_i : \Omega \longrightarrow \mathbb{C},$$

has the following properties:

(1) $\sum g_{ij}\phi_i\overline{\phi_j} \neq 0$,
(2) $\sum g_{ij}\phi_i\phi_j = 0$,
(3) $\dfrac{\partial \phi_i}{\partial \overline{z}} + \sum_{k,l} \Gamma^i_{kl}\phi_k\overline{\phi_l} = 0.$

where the $\{\Gamma^i_{kl}\}$ are the Christoffel symbols of the Riemannian connection.

Conversely, given functions $\phi_i : \Omega \longrightarrow \mathbb{C}$ that verify the above conditions, the differential ϕdz has no real periods and the map:

$$f:\Omega\longrightarrow M, \qquad f(z) = 2\,\mathcal{R}e\int_{z_0}^{z}\phi\, dz,$$

is a conformal minimal immersion of Ω into M (here z_0 is a arbitrary fixed point of Ω).

Remark 2.2. The first condition tells us that f is an immersion, the second that is conformal and the last one that f is minimal. The last condition is called the *holomorphicity condition* since is the local coordinates version of the condition: $\tilde{\nabla}_{\frac{\partial}{\partial \overline{z}}}\phi = 0$, where $\tilde{\nabla}$ is the induced connection on the pull-back bundle $f^*(TM\otimes\mathbb{C})$.

In general is quite difficult to produce functions ϕ_i with the above properties since the holomorphicity condition is a non linear P.D.E. with *nonconstant coefficients*. If M is a Lie group equipped with a left invariant metric g and E_i are ortonormal left invariant fields, we can write

$$\phi = \sum \phi_i \frac{\partial}{\partial x_i} = \sum \psi_i E_i, \qquad \psi_i : \Omega \longrightarrow \mathbb{C},$$

with $\phi_i = \sum A_{ij}\psi_j$ and $A = (A_{ij})$ being an invertible functions valued matrix. In this case the Weierstrass formula becomes:

Theorem 2.3. *Given functions $\psi_i:\ \Omega\longrightarrow\mathbb{C}$ such that:*

(1) $\sum_i |\psi_i|^2 \neq 0$,
(2) $\sum_i \psi_i^2 = 0$,
(3) $\dfrac{\partial \psi_i}{\partial \overline{z}} + \sum_{j,k} L^i_{jk}\overline{\psi_j}\psi_k = 0,$

where $L^i_{jk} := g(\nabla_{E_j}E_k, E_i)$, then the map:

$$f:\Omega\longrightarrow M, \qquad f_i(z) = 2\,\mathcal{R}e\Big(\int_{z_0}^{z}\sum_j A_{ij}\psi_j\, dz\Big),$$

defines a conformal minimal immersion.

The advantage of having a constant coefficients P.D.E. is not really a great gain, in principle, since we still have to compute the integrand $A_{ij}\psi_j$ along the solutions. However, in certain cases, as, for example, the hyperbolic space, the Heisenberg group and the product of the hyperbolic plane with $\mathbb{R}$, this problem may be overcome by ad hoc arguments, as shown, for example, in ([2]).

3. The Björling problem for tridimensional Lie group

In this section will suppose that M is a tridimensional Lie group endowed with a left invariant Riemannian metric g. Let $\beta : I \subseteq \mathbb{R} \to M$ be a regular analytic curve in M and $V : I \to TM$ an unitary real analytic vector field along β, such that $g(\dot\beta, V) \equiv 0$. The Björling problem is the following:

Determine a minimal surface $f : I \times (-\epsilon, \epsilon) = \Omega \subseteq \mathbb{C} \longrightarrow M$, such that:

- $f(u, 0) = \beta(u)$,
- $N(u, 0) = V(u)$,

for all $u \in I$, where $N : \Omega \to TM$ the Gauss map of the surface.

We observe that if β is parameterized by arc length and $\ddot\beta := \nabla_{\dot\beta}\dot\beta$, we have that $V = \|\ddot\beta\|^{-1}\ddot\beta$ is a unit vector field along the curve such that $g(\dot\beta, V) \equiv 0$. Then the Björling problem is a generalization of the problem of finding a minimal surface which contains a given curve as a geodesic.

Theorem 3.1. *The Björling problem has an unique solution*[1].

Proof. In order to prove the theorem we must analise Theorem (2.3) a bit more. In (2.3) we have essentially four conditions on the three functions ψ_i (the first condition is "generically satisfied"). We will start showing that these conditions are dependent.

Lemma 3.2. *Let $\psi_i : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$, $i = 1, 2$, be two differentiable functions and $\psi_3^2 = -\psi_1^2 - \psi_2^2$. We suppose that ψ_i, $i = 1, 2$, satisfy the two first equations of* (2.3, (3)). *Then ψ_3 satisfies the third equation.*

Proof. Deriving with respect to $\bar z$ the equation

$$-\psi_3^2 = (\psi_1^2 + \psi_2^2),$$

and using the fact that the two first equations of (2.3, (3)) are satisfied, we have:

$$\begin{aligned} -\psi_3 \frac{\partial\psi_3}{\partial\bar z} &= \psi_1 \frac{\partial\psi_1}{\partial\bar z} + \psi_2 \frac{\partial\psi_2}{\partial\bar z} \\ &= -\sum_{j,k=1}^{3} (L_{jk}^1\,\psi_1 + L_{jk}^2\,\psi_2)\,\bar\psi_j\,\psi_k. \end{aligned}$$

[1]Unique up to fixing the domain.

Therefore, to prove the proposition it suffices to show that

$$\sum_{j,k=1}^{3} (L^1_{jk}\,\psi_1 + L^2_{jk}\,\psi_2 + L^3_{jk}\,\psi_3)\bar{\psi}_j\,\psi_k = 0.$$

Writing the above sum as:

$$\sum_{j,k=1}^{3} L^k_{jk}\,\bar{\psi}_j\,\psi_k^2 + \sum_{\substack{j,\,k,\,l\,=\,1\\ k\,<\,l}}^{3} (L^l_{jk} + L^k_{jl})\,\bar{\psi}_j\,\psi_k\,\psi_l,$$

and using the relations $L^l_{jk} + L^k_{jl} = 0, \forall\, j,k,l \in \{1,2,3\}$, we conclude the proof. $\square$

We go back now to the proof of the Theorem. Consider the system:

$$\begin{cases} \dfrac{\partial\psi_1}{\partial\overline{z}} + \displaystyle\sum_{j,k=1}^{3} L^1_{jk}\overline{\psi_j}\psi_k = 0, \\ \dfrac{\partial\psi_2}{\partial\overline{z}} + \displaystyle\sum_{j,k=1}^{3} L^2_{jk}\overline{\psi_j}\psi_k = 0, \end{cases}$$

where $\psi_i : \Omega \longrightarrow \mathbb{C}$, and $\psi_3 = [-\psi_1^2 - \psi_2^2]^{\frac{1}{2}}$.

Since this system is of Cauchy-Kovalevskaya type, fixing the initial datas $\psi_i(u,0)$, $i = 1,2$, it has, locally, a unique solution. This solution gives, via (2.3) and (3.2), a minimal surface. Thus we must find initial conditions guarantying that this surface has the required properties. Observe that, if f is a solution of the Björling problem, we have

$$\phi(u,0) := \frac{1}{2}\Big(\frac{\partial f}{\partial u} - i\frac{\partial f}{\partial v}\Big)(u,0) = \frac{1}{2}\big(\dot{\beta}(u) + i\,\dot{\beta}(u)\wedge V(u)\big).$$

Therefore the initial data for the system is:

$$\psi(u,0) = A^{-1}(\beta(u))\phi(u,0).$$

We observe that the initial conditions imply $\dfrac{\partial f}{\partial u}(u,0) = \dot{\beta}(u)$. Hence, up to a constant determined by the constant of integration in (2.3), we have $f(u,0) = \beta(u)$. Also the initial condition forces the choice of one of the determinations of $\psi_3^2 = -\psi_1^2 - \psi_2^2$.

Up to now we have proved the existence of a local solution to the problem. Using compactness of I and local uniqueness, we have existence and uniqueness of the solution when $\beta(I)$ is contained in a coordinate neighborhood, for ϵ sufficiently small. Covering I with a finite number of inverse images, via β, of coordinate neighborhoods and using again the uniqueness of the local problem, the result is proved for the general case. $\square$

References

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